Inverse energy cascade in nonlocal helical shell models of turbulence

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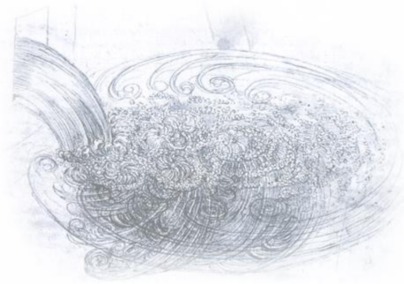
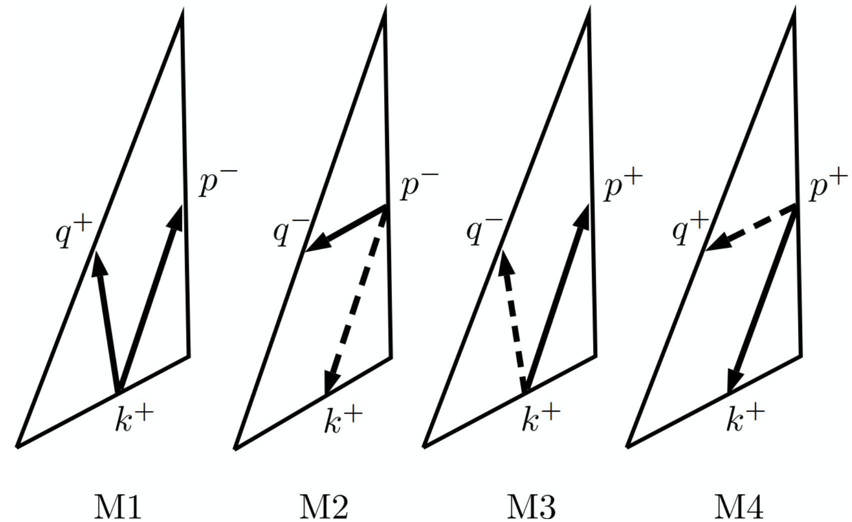


Introduction

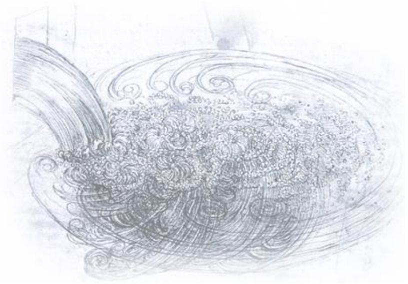
- **Helicity** and its role in turbulence
- **Helical decomposition** for Navier Stokes
- NS dynamics is a **combination of many interactions** with different transfer properties

Non-linear interactions between triads of wavenumbers in NS



  - ➢ in 3D turbulence there are interactions with a **direct or inverse energy cascade**

- **Shell-models** represent the best approach (so far) to study these different interactions



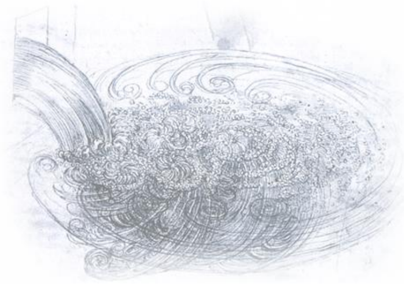
Helicity and turbulence

In 3D NS equations there exist two inviscid invariants: **Energy** and **Helicity**

$$E = \int d\mathbf{x}\, |\mathbf{v}|^2 \qquad\qquad H = \int d\mathbf{x}\, \mathbf{v}\cdot\boldsymbol{\omega}, \qquad \boldsymbol{\omega} = \nabla\times\mathbf{v}$$

Helicity:

- **Pseudoscalar**: sensitive to **parity breaking**.
- **Not sign definite**, in principle it should **not block forward energy cascade**.
- Significant for geophysical and astrophysical/MHD flows.



Helical decomposition of Navier Stokes equations

Incompressible NS equations in Fourier space:

$$\partial_t u_j(\mathbf{k},t) = -ik_m P_{jl}(\mathbf{k}) \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}=0} u_l(\mathbf{p},t) u_m(\mathbf{q},t) + F(\mathbf{k},t) - \nu k^2 u_j(\mathbf{k},t)$$

Waleffe* introduces orthonormal basis: $\boldsymbol{\kappa}(\mathbf{k}), \mathbf{h}^+(\mathbf{k}), \mathbf{h}^-(\mathbf{k})$

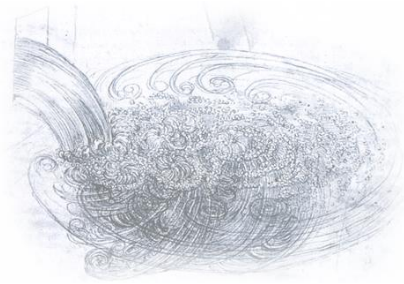
where: $\boldsymbol{\kappa} = \frac{\mathbf{k}}{|\mathbf{k}|}$; and $\mathbf{h}^+, \mathbf{h}^-$ are eigenvectors of the curl operator

Velocity becomes: $\mathbf{v}(\mathbf{x}) = \sum_{\mathbf{k}} \mathbf{u}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_{\mathbf{k}} (u^+_{\mathbf{k}} \mathbf{h}^+_{\mathbf{k}} + u^-_{\mathbf{k}} \mathbf{h}^-_{\mathbf{k}}) e^{i\mathbf{k}\cdot\mathbf{x}}$

Energy: $E = \sum_{\mathbf{k}} (|u^+_{\mathbf{k}}|^2 + |u^-_{\mathbf{k}}|^2)$

Helicity: $H = \sum_{\mathbf{k}} k(|u^+_{\mathbf{k}}|^2 - |u^-_{\mathbf{k}}|^2)$

*Waleffe, *The nature of triad interactions in homogeneous turbulence*, Phys. Fluids A, 1992

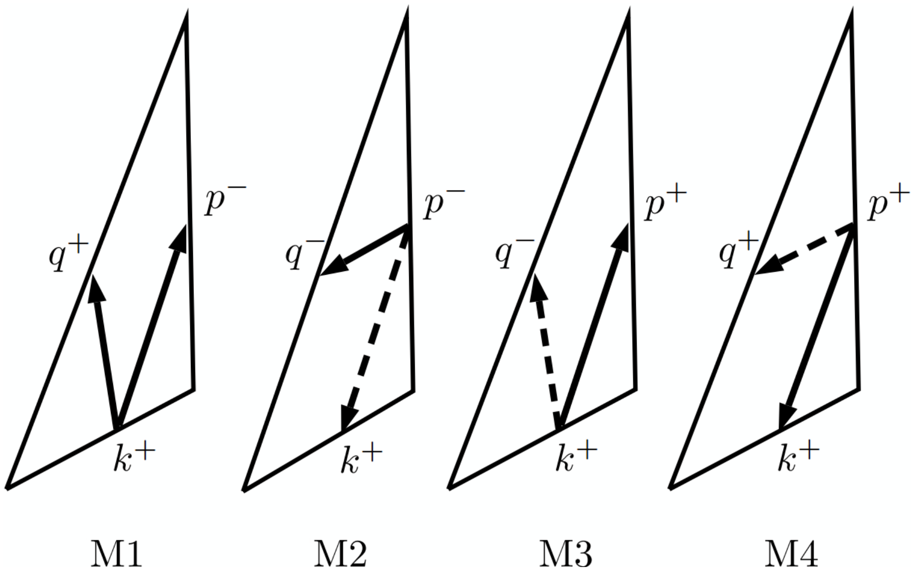


Helical decomposition of Navier Stokes equations

Time evolution of a helical Fourier mode:

$$\partial_t u_{\mathbf{k}}^{\pm}(t) = -\frac{1}{4} \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}=0} \mathbf{g}_{\mathbf{k},\mathbf{p},\mathbf{q}} u_{\mathbf{p}}^{\pm *}(t) u_{\mathbf{q}}^{\pm *}(t) + F_{\mathbf{k}}^{\pm}(t) - \nu k^2 u_{\mathbf{k}}^{\pm}(t)$$

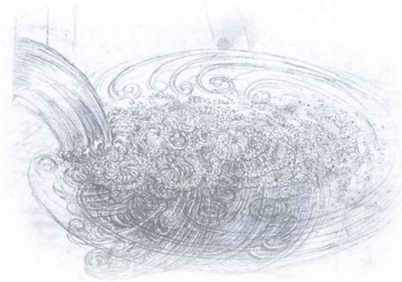
There are only 4 independent helical triadic interactions



Question: what happens if we restrict the NS dynamics to only one subclass of interactions?

In a simulation, one needs to select explicitly the triads to evolve.

- **Spectral DNS** of Navier Stokes → Low Reynolds number, **problem**!
- **Shell models of turbulence are the only possible approach**



Shell models for turbulence

Shell models are dynamical systems inspired by the NS equations.

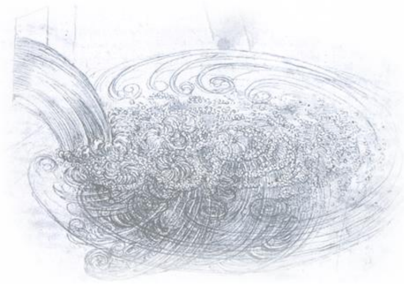
Features of **helical SABRA shell models**:

1) System of 2N one-dimensional equations, two complex variables $u_n^{\pm}$ per shell, representing the NS velocity fluctuation
2) Discrete, logarithmically spaced shells in Fourier space: $k_n = k_0 \lambda^n$, with $\lambda > 1$
3) **Triadic interactions**, as in NS
4) **First neighbor interactions** $(\mathbf{u_n}, \mathbf{u_{n+1}}, \mathbf{u_{n+2}})$
5) **Energy and helicity conserved triad by triad**, as in NS

Equations of the model(s)*:

$$(\partial_t + \nu k_n^2) u_n^{+} = i(a k_{n+1} u_{n+2}^{\pm} u_{n+1}^{\pm *} + b k_n u_{n+1}^{\pm} u_{n-1}^{\pm *} + c k_{n-1} u_{n-1}^{\pm} u_{n-2}^{\pm}) + f_n^{+}$$
$$(\partial_t + \nu k_n^2) u_n^{-} = i(a k_{n+1} u_{n+2}^{\mp} u_{n+1}^{\mp *} + b k_n u_{n+1}^{\mp} u_{n-1}^{\mp *} + c k_{n-1} u_{n-1}^{\mp} u_{n-2}^{\mp}) + f_n^{-}$$

*Benzi et al, *Helical shell models for three-dimensional turbulence*, PRE, 1996

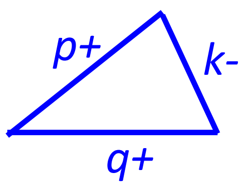


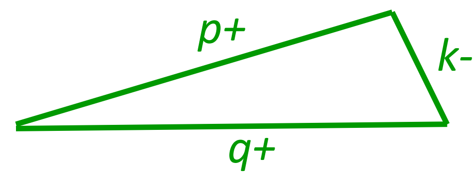
Shell model with elongated triads

The **geometry of the triad** can be a crucial factor for the dynamics of the system (both in NS and shell models)*

For one class of helical interaction (M2), **energy flux** can reverse its direction **depending on the triad geometry**.

M2
$k/p > 0{,}278$





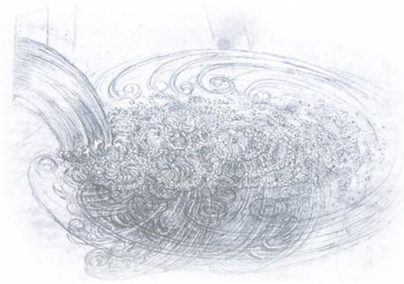
M2 (elongated)
$k/p < 0{,}278$

We introduce a different version of the helical SABRA shell model for interaction M2, with *elongated* triads, which is expected to show an inverse energy cascade.

Equations of the model (**M2 elongated**): $(\mathbf{u_n}, \mathbf{u_{n+2}}, \mathbf{u_{n+3}})$

$$(\partial_t + \nu k_n^2) u_n^+ = i(a k_{n+2} u_{n+3}^- u_{n+2}^{+*} + b k_n u_{n+1}^+ u_{n-2}^{-*} + c k_{n-1} u_{n-1}^+ u_{n-3}^-) + f_n^+$$
$$(\partial_t + \nu k_n^2) u_n^- = i(a k_{n+2} u_{n+3}^+ u_{n+2}^{+*} + b k_n u_{n+1}^- u_{n-2}^{+*} + c k_{n-1} u_{n-1}^- u_{n-2}^+) + f_n^-$$

*Waleffe, *The nature of triad interactions in homogeneous turbulence*, Phys. Fluids A, 1992



Energy flux direction

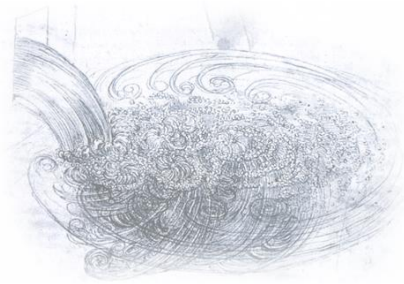
Prediction for the energy flux direction *a la Waleffe*, based on the **linear stability analysis** of a **single triad:**

Stationary state, constant energy flux: $\langle \Pi_n^E \rangle = \underline{f(b)} \langle \delta_n^E \rangle = const$

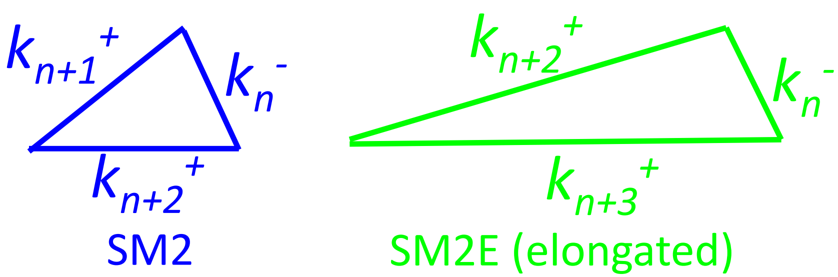
$$\delta_n^E = -2 k_n Im[(u_{n+1}^{s_3} u_n^{+*} u_{n-1}^{s_4 *}) + (u_{n+1}^{-s_3} u_n^{-*} u_{n-1}^{-s_4 *})]$$

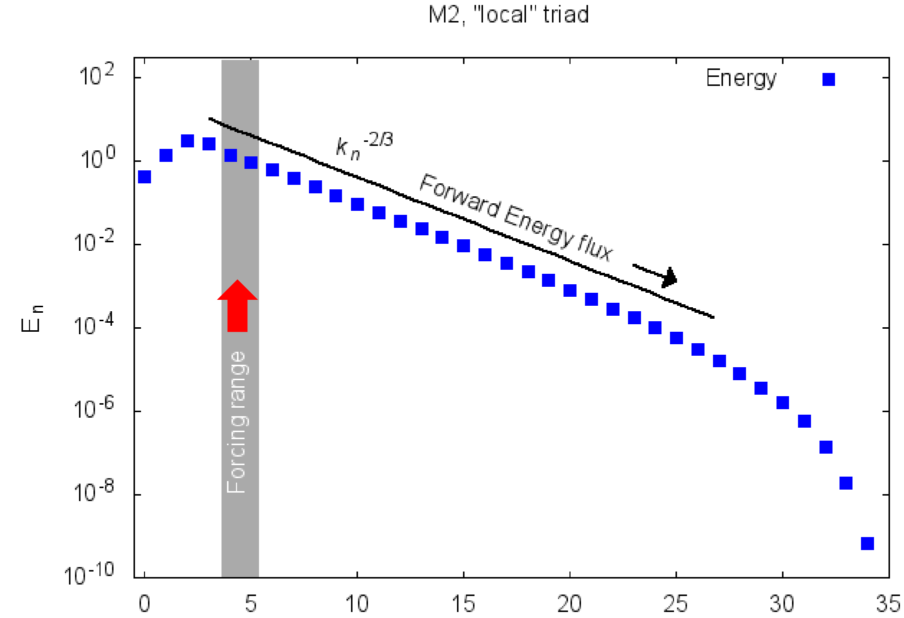
- **f(b) changes sign depending on the triad geometry**
- **$\langle \delta^E \rangle$ stays constant**

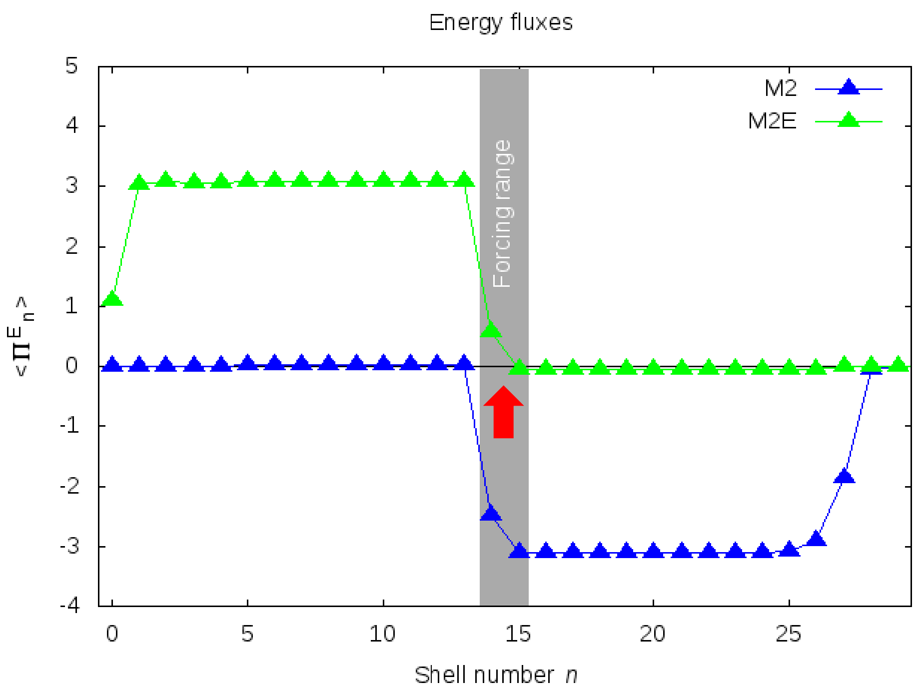
→ Transition in the energy flux direction, depending on the triad geometry

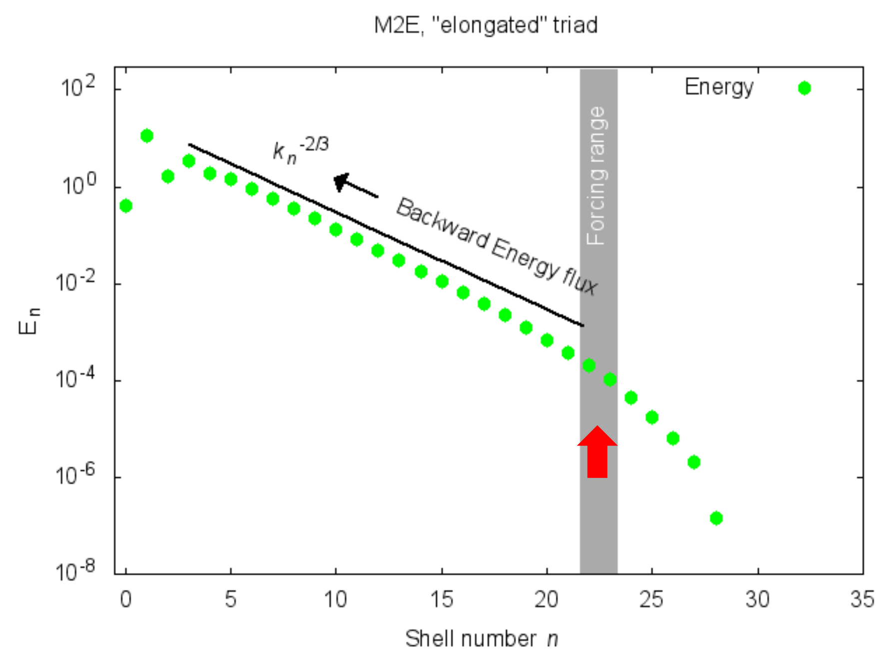


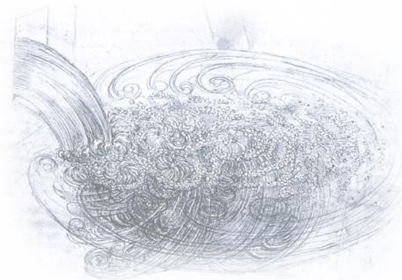
Numerical simulations











Conclusions

- We developed a shell model for 3D turbulence which conserves both Energy and Helicity (in the inviscid limit) and that shows a transition from direct to inverse cascade at changing the triad geometry
- In agreement with a phenomenological argument given by Waleffe for the corresponding sub-set of triads in the real NS equations
- Future directions: studying the scaling and intermittency properties of energy/helicity flux in both regimes and/or at combining models with different transfer properties

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