Inverse energy cascade in nonlocal helical shell models of turbulence

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Introduction

- **Helicity** and its role in turbulence
- **Helical decomposition** for Navier Stokes
- NS dynamics is a **combination of many interactions** with different transfer properties

> in 3D turbulence there are interactions with a **direct or inverse energy cascade**

- **Shell-models** represent the best approach (so far) to study these different interactions
Helicity and turbulence

In 3D NS equations there exist two inviscid invariants: **Energy** and **Helicity**

\[
E = \int d\mathbf{x} |\mathbf{v}|^2 \quad \quad \quad H = \int d\mathbf{x} \mathbf{v} \cdot \mathbf{\omega}, \quad \mathbf{\omega} = \nabla \times \mathbf{v}
\]

**Helicity:**
- **Pseudoscalar**: sensitive to parity breaking.
- **Not sign definite**, in principle it should **not block forward energy cascade**.
- Significant for geophysical and astrophysical/MHD flows.
Helical decomposition of Navier Stokes equations

Incompressible NS equations in Fourier space:

$$\partial_t u_j(k, t) = -i k_m P_{jl}(k) \sum_{k+p+q=0} u_l(p, t) u_m(q, t) + F(k, t) - \nu k^2 u_j(k, t)$$

Waleffe* introduces orthonormal basis: $\kappa(k), h^+(k), h^-(k)$

where: $\kappa = \frac{k}{|k|}$; and $h^+, h^-$ are eigenvectors of the curl operator

Velocity becomes: $v(x) = \sum_k u(k) e^{ik \cdot x} = \sum_k (u^+_k h^+_k + u^-_k h^-_k) e^{ik \cdot x}$

Energy: $E = \sum_k (|u^+_k|^2 + |u^-_k|^2)$

Helicity: $H = \sum_k k(|u^+_k|^2 - |u^-_k|^2)$

Helical decomposition of Navier Stokes equations

Time evolution of a helical Fourier mode:

\[ \partial_t u_k^\pm(t) = -\frac{1}{4} \sum_{k+p+q=0} g_{k,p,q} u_p^{\pm*}(t) u_q^{\pm*}(t) + F_k^\pm(t) - \nu k^2 u_k^\pm(t) \]

There are only 4 independent helical triadic interactions

**Question:** what happens if we restrict the NS dynamics to only one subclass of interactions?

In a simulation, one needs to select explicitly the triads to evolve.

- **Spectral DNS** of Navier Stokes → Low Reynolds number, **problem**!
- **Shell models of turbulence are the only possible approach**
Shell models for turbulence

Shell models are dynamical systems inspired by the NS equations.

Features of helical SABRA shell models:

1) System of $2N$ one-dimensional equations, two complex variables $u_{n}^{±}$ per shell, representing the NS velocity fluctuation

2) Discrete, logarithmically spaced shells in Fourier space: $k_{n} = k_{0} \lambda^{n}$, with $\lambda > 1$

3) Triadic interactions, as in NS

4) First neighbor interactions $(u_{n}, u_{n+1}, u_{n+2})$

5) Energy and helicity conserved triad by triad, as in NS

Equations of the model(s)*:

$$(\partial_{t} + \nu k_{n}^{2}) u_{n}^{+} = i(ak_{n+1} u_{n+2}^{±} u_{n+1}^{±*} + bk_{n} u_{n+1}^{±} u_{n-1}^{±*} + ck_{n-1} u_{n-1}^{±} u_{n-2}^{±}) + f_{n}^{+}$$

$$(\partial_{t} + \nu k_{n}^{2}) u_{n}^{-} = i(ak_{n+1} u_{n+2}^{±} u_{n+1}^{±*} + bk_{n} u_{n+1}^{±} u_{n-1}^{±*} + ck_{n-1} u_{n-1}^{±} u_{n-2}^{±}) + f_{n}^{-}$$

*Benzi et al, *Helical shell models for three-dimensional turbulence*, PRE, 1996
The geometry of the triad can be a crucial factor for the dynamics of the system (both in NS and shell models)*

For one class of helical interaction (M2), energy flux can reverse its direction depending on the triad geometry.

We introduce a different version of the helical SABRA shell model for interaction M2, with elongated triads, which is expected to show an inverse energy cascade.

Equations of the model (M2 elongated): \((u_n, u_{n+2}, u_{n+3})\)

\[
(\partial_t + \nu k_n^2)u_n^+ = i(ak_{n+2}u_{n+3}^+ u_{n+2}^- + bk_n u_{n+1}^+ u_{n-2}^- + ck_{n-1} u_{n-1}^+ u_{n-3}^-) + f_n^+ \\
(\partial_t + \nu k_n^2)u_n^- = i(ak_{n+2}u_{n+3}^- u_{n+2}^+ + bk_n u_{n+1}^- u_{n-2}^+ + ck_{n-1} u_{n-1}^- u_{n-2}^+) + f_n^-
\]

Energy flux direction

Prediction for the energy flux direction *a la Waleffe*, based on the linear stability analysis of a single triad:

Stationary state, constant energy flux: \[ \langle \Pi_n^E \rangle = f(b) \langle \delta_n^E \rangle = \text{const} \]

\[ \delta_n^E = -2k_n \text{Im} \left[ (u_{n+1}^{s3} u_n^{4*} u_{n-1}^{s4*}) + (u_{n+1}^{s3} u_n^{-4*} u_{n-1}^{-s4*}) \right] \]

- \( f(b) \) changes sign depending on the triad geometry
- \( \langle \delta^E \rangle \) stays constant

→ Transition in the energy flux direction, depending on the triad geometry
Numerical simulations

\[ k_{n+1}^+ k_n^- k_{n+2}^+ \]

\[ k_{n+2}^+ k_n^+ k_{n+3}^- \]

SM2

SM2E (elongated)

\[ k_n - k_{n+1} + k_{n+2} + k_{n+3} \]
Conclusions

- We developed a shell model for 3D turbulence which conserves both Energy and Helicity (in the inviscid limit) and that shows a transition from direct to inverse cascade at changing the triad geometry.

- In agreement with a phenomenological argument given by Waleffe for the corresponding sub-set of triads in the real NS equations.

- Future directions: studying the scaling and intermittency properties of energy/helicity flux in both regimes and/or at combining models with different transfer properties.

References:

- Biferale, Musacchio, Toschi, *Inverse energy cascade in 3D isotropic turbulence*, PRL, 2012
- Benzi, Biferale, Kerr, Trovatore, *Helical shell models for three-dimensional turbulence*, PRE, 1996