Tumbling of small non-spherical particles in a shear flow

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Effect of weak fluid inertia on Jeffery orbits,
Shear flow

Shear flow $u(y) = s y \hat{x}$, flow-gradient matrix $A = \begin{bmatrix} 0 & s & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Degenerate orientational dynamics of small axisymmetric particles: Jeffery orbits (tumbling).


Expect that perturbations have strong effect.

Although this limit may seem to be of limited interest, ...

... these small changes can have a strong (or even dominant) cumulative effect on the particle's position or orientation. This occurs for the class of so-called “indeterminate” particle motions, in which no position or orientation is intrinsically favored under “standard conditions.”

L. G. Leal
Tumbling in simple shear


Micron-sized glass rods in a micro-channel flow. $n_z = 1$: log-rolling, $n_z = 0$: tumbling. Symmetry axis $n$ spends a long time aligned with flow-direction $\hat{x}$.

$n$-dynamics

Fig. 3a. Einarsson et al 2015. Scale real time.

$n_x$ (blue solid), $n_z$ (red dashed). Forward, dashed back.
Dimensionless parameters

Axisymmetric particles: shape parameter \( \Lambda = \frac{(\lambda^2 - 1)}{(\lambda^2 + 1)} \)

(aspect ratio \( \lambda = \frac{a}{c} \)).

Fluid inertia: \( \text{Re}_s = \frac{a^2 s}{\nu} \)

shear Reynolds number

\( \nu \) kinematic viscosity

\( s \) shear rate

Particle inertia: \( \text{St} = \frac{\rho_p}{\rho_f} \text{Re}_s \)

Stokes number

\( \rho_p \) particle density

\( \rho_f \) fluid density

Brownian rotation: \( \text{Pe} = \frac{s}{D} \)

Péclet number

where \( D \) rotational diffusion constant

Jeffery equation obtained for small particles for \( \text{Re}_s = 0, \text{St} = 0, \text{Pe} = \infty \).

Degeneracy: must consider effect of perturbations. Here: \( \text{Re}_s > 0 \).
Inertial effects - particles in flows

Particle- and Navier Stokes equations coupled by boundary conditions.

Maxey-Riley equation, correction to Stokes law

Saffman lift on small sphere due to shear.

Tumbling of a neutrally buoyant fibre in shear flow, slender-body limit.

Log-rolling unstable.

Tumbling of a nearly spherical neutrally buoyant particle in shear.

Log-rolling stable.
Effective equation of motion

Neutrally buoyant spheroid.

Find effective vector field, correction to Jeffery’s equation (caveat: caustics)

\[ \dot{n} = F_0(n) + StF_1(n) + Re_s F_2(n) + \ldots \]

\( F_1(n) \) known. Einarsson, Angilella & Mehlig, Physica D 278-279, 79 (2014)

Now \( F_2(n) \). Difficulty: need to calculate torque on particle.

Requires solving Navier Stokes equations.

Perturbation theory in \( Re_s, St \), neglect terms of order \( St Re_s, St^2, Re_s^2, \ldots \).

Still very difficult problem. Solve it by exploiting the symmetries of problem.
Equations of motion

Orientational motion of neutrally buoyant spheroid

\[ \dot{n}_i = \varepsilon_{ijk} \omega_j n_k, \quad \text{St}(I_{ij} \dot{\omega}_j + \dot{I}_{ij} \omega_j) = T_i \]

Angular velocity \( \omega_j \), particle inertia tensor \( I_{ij} = A^I(\delta_{ij} - P_{ij}) + B^I P_{ij} \) with \( P_{ij} = \delta_{ij} - n_in_j \), \( A^I \) and \( B^I \) are moments of inertia along and \( \perp \) to \( n \).

Hydrodynamic torque \( T_i \) determined by integrating fluid stresses over particle surface \( S \). Requires solving Navier Stokes equations. Dimensionless variables (scales: time \( T = s^{-1} \), length \( L = a \), velocity \( U = sa \), pressure \( P = \mu s \) ):

\[
\text{Re}_s(\partial_t u_i + u_j \partial_j u_i) = -\partial_i p + \partial_j \partial_j u_i
\]

\( \partial_i u_i = 0 \), \( u_i = \varepsilon_{ijk} \omega_j r_k \) for \( r \in S \),

\[ u_i = u_i^{(\infty)} \text{ as } r \to \infty. \]

\( p \) pressure, \( \omega \) particle angular velocity, \( u_i^{(\infty)} \) shear in dimensionless variables.
Perturbation theory

Use 'reciprocal theorem' to compute hydrodynamic torque.

Result (simple shear flow)

\[ T_k = T_k^{(0)} - \text{Re}_s \int_V dv \tilde{U}_{ik} (\partial_t u_i + u_j \partial_j u_i) \]

The coefficients \( \tilde{U}_{ik} \) are obtained by solving auxiliary Stokes problem. Above relation is exact. But unknown \( u_i \), solution of Navier Stokes equations.

Perturbation theory: \( St = Re_s = 0 \) - solutions to evaluate \( T_k \) to leading order.

Then expand:

\[ \omega_i = \omega_i^{(0)} + St \omega_i^{(St)} + Re_s \omega_i^{(Re_s)} + \ldots . \]

Insert into angular-momentum equation. Find: \( \omega_i^{(0)} \) Jeffery angular velocity, \( \omega_i^{(St)} \) particle-inertia contribution, \( \omega_i^{(Re_s)} \) fluid-inertia contribution, depends on volume integral. Difficult to calculate. Integrand depends non-linearly on \( n \).

Symmetries

Make use of the symmetries of the problem. Simple shear flow $A^\infty = S^\infty + O^\infty$.

Symmetries constrain form of equation of motion.

\[
\dot{n}_i = O_{i q} n_q + \Lambda [S_{i p} n_p - (n_p S_{p q} n_q) n_i] \\
+ \beta_1 (n_p S_{p q} n_q) P_{i j} S_{j k} n_k + \beta_2 (n_p S_{p q} n_q) O_{i j} n_j \\
+ \beta_3 P_{i j} O_{j k} S_{k l} n_l + \beta_4 P_{i j} S_{j k} S_{k l} n_l .
\]

First row: Jeffery equation. Remainder: $St$- and $Res$-corrections, determined by only four scalar functions

\[
\beta_\alpha = St \beta_{\alpha}^{(St)}(\lambda) + Res \beta_{\alpha}^{(Res)}(\lambda) \quad \text{for } \alpha = 1, \ldots, 4 .
\]

Can be calculated.
Results

Stability of log-rolling and tumbling orbits.

Linear stability analysis at small $\text{Re}_s$.

Stability exponents $\gamma_{LR}$, $\gamma_T$
Conclusions

Orientational dynamics of neutrally buoyant axisymmetric particle in shear. Stability analysis of Jeffery orbits at infinitesimal $Re_s$. Results:

-log-rolling unstable for prolate particles, tumbling in shear plane stable.
-for oblate particles (but not too disk-like) stabilities are reversed
-fluid inertia contributes more strongly than particle inertia
-both unsteady and convective fluid inertia matter. It would be qualitatively wrong to neglect either.

To do:
-analyse orientational motion for small but finite $Re_s$. Rosén, Lundell & Aidun (2014)
-wall effects
-settling $\rho_p \neq \rho_f$ (more difficult)
-unsteady flows (more difficult)
-turbulence (much more difficult)